

Some conjectures on congruences

ZHI-HONG SUN

School of Mathematical Sciences, Huaiyin Normal University,
Huaian, Jiangsu 223001, PR China
Email: zhihongsun@yahoo.com
Homepage: <http://www.hytc.edu.cn/xsjl/szh>

ABSTRACT. Let p be an odd prime. In the paper we collect the author's various conjectures on congruences modulo p or p^2 , which are concerned with sums of binomial coefficients, Lucas sequences, power residues and special binary quadratic forms.

MSC: Primary 11A07, Secondary 05A10, 11B39, 11E25,
Keywords: congruence; binomial coefficient; Lucas sequence; binary quadratic form

1. Notation.

Let \mathbb{Z} and \mathbb{N} be the sets of integers and positive integers respectively. For $b, c \in \mathbb{Z}$ the Lucas sequences $\{U_n(b, c)\}$ and $\{V_n(b, c)\}$ are defined by

$$(1.1) \quad \begin{aligned} U_0(b, c) &= 0, \quad U_1(b, c) = 1, \\ U_{n+1}(b, c) &= bU_n(b, c) - cU_{n-1}(b, c) \quad (n \geq 1) \end{aligned}$$

and

$$(1.2) \quad \begin{aligned} V_0(b, c) &= 2, \quad V_1(b, c) = b, \\ V_{n+1}(b, c) &= bV_n(b, c) - cV_{n-1}(b, c) \quad (n \geq 1). \end{aligned}$$

Let $d = b^2 - 4c$. It is well known that for $n \in \mathbb{N}$,

$$(1.3) \quad U_n(b, c) = \begin{cases} \frac{1}{\sqrt{d}} \left\{ \left(\frac{b+\sqrt{d}}{2} \right)^n - \left(\frac{b-\sqrt{d}}{2} \right)^n \right\} & \text{if } d \neq 0, \\ n \left(\frac{b}{2} \right)^{n-1} & \text{if } d = 0 \end{cases}$$

and

$$(1.4) \quad V_n(b, c) = \left(\frac{b+\sqrt{d}}{2} \right)^n + \left(\frac{b-\sqrt{d}}{2} \right)^n.$$

1

Let $[x]$ be the greatest integer not exceeding x , and let $(\frac{a}{m})$ be the Jacobi symbol. For $m \in \mathbb{Z}$ with $m = 2^\alpha m_0$ ($2 \nmid m_0$) we say that $2^\alpha \parallel m$ and m_0 is the odd part of m . For $t \in \mathbb{Z}$ let

$$\delta(t) = \begin{cases} 1 & \text{if } 8 \mid t, \\ -1 & \text{if } 8 \nmid t. \end{cases}$$

For an integer m and odd prime p with $p \nmid m$ let

$$Z_p(m) = \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{m^n} \sum_{k=0}^n \binom{n}{k}^3.$$

2. Conjectures on power residues.

In 1980 and 1984 Hudson and Williams proved the following result.

Theorem 2.1. *Let $p \equiv 1 \pmod{24}$ be a prime and hence $p = c^2 + d^2 = x^2 + 3y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$.*

- (i) ([HW]) *If $c \equiv \pm(-1)^{\frac{y}{4}} \pmod{3}$, then $3^{\frac{p-1}{8}} \equiv \pm 1 \pmod{p}$.*
- (ii) ([H]) *If $d \equiv \pm(-1)^{\frac{y}{4}} \pmod{3}$, then $3^{\frac{p-1}{8}} \equiv \pm \frac{d}{c} \pmod{p}$.*

Hudson and Williams proved Theorem 2.1(i) by using the cyclotomic numbers of order 12, and Hudson proved Theorem 2.1(ii) using the Jacobi sums of order 24.

In [S3] the author posed the following conjectures similar to Theorem 2.1.

Conjecture 2.1 ([S3, Conjecture 9.1]). *Let $p \equiv 13 \pmod{24}$ be a prime and hence $p = c^2 + d^2 = x^2 + 3y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$, $x = 2^\alpha x_0$, $y = 2^\beta y_0$ and $x_0 \equiv y_0 \equiv 1 \pmod{4}$. Then*

$$3^{\frac{p-5}{8}} \equiv \begin{cases} \pm \frac{y}{x} \pmod{p} & \text{if } x \equiv \pm c \pmod{3}, \\ \mp \frac{dy}{cx} \pmod{p} & \text{if } x \equiv \pm d \pmod{3}. \end{cases}$$

Conjecture 2.1 has been checked for all primes $p < 20,000$.

Conjecture 2.2 ([S3, Conjecture 9.2]). *Let $p \equiv 1, 9, 25 \pmod{28}$ be a prime and hence $p = c^2 + d^2 = x^2 + 7y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$, $x = 2^\alpha x_0$, $y = 2^\beta y_0$ and $x_0 \equiv y_0 \equiv 1 \pmod{4}$.*

- (i) *If $p \equiv 1 \pmod{8}$, then*

$$7^{\frac{p-1}{8}} \equiv \begin{cases} -(-1)^{\frac{y}{4}} \pmod{p} & \text{if } 7 \mid c, \\ (-1)^{\frac{y}{4}} \pmod{p} & \text{if } 7 \mid d, \\ \mp (-1)^{\frac{y}{4}} \frac{d}{c} \pmod{p} & \text{if } c \equiv \pm d \pmod{7}. \end{cases}$$

- (ii) *If $p \equiv 5 \pmod{8}$, then*

$$7^{\frac{p-5}{8}} \equiv \begin{cases} -\frac{y}{x} \pmod{p} & \text{if } 7 \mid c, \\ \frac{y}{x} \pmod{p} & \text{if } 7 \mid d, \\ \mp \frac{dy}{cx} \pmod{p} & \text{if } c \equiv \pm d \pmod{7}. \end{cases}$$

Conjecture 2.2 has been checked for all primes $p < 20,000$.

Conjecture 2.3 ([S3, Conjecture 9.7]). *Let $p \equiv 1, 9 \pmod{20}$ be a prime and hence $p = c^2 + d^2 = x^2 + 5y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$ and all the odd parts of d, x, y are congruent to 1 modulo 4.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$5^{\frac{p-1}{8}} \equiv \begin{cases} \pm(-1)^{\frac{d}{4}}\delta(y) \pmod{p} & \text{if } x \equiv \pm c \pmod{5}, \\ \pm(-1)^{\frac{d}{4}}\delta(y)\frac{d}{c} \pmod{p} & \text{if } x \equiv \pm d \pmod{5}. \end{cases}$$

(ii) *If $p \equiv 5 \pmod{8}$, then*

$$5^{\frac{p-5}{8}} \equiv \begin{cases} \pm\delta(x)\frac{dy}{cx} \pmod{p} & \text{if } x \equiv \pm c \pmod{5}, \\ \mp\delta(x)\frac{y}{x} \pmod{p} & \text{if } x \equiv \pm d \pmod{5}. \end{cases}$$

Conjecture 2.3 has been checked for all primes $p < 20,000$.

Conjecture 2.4 ([S3, Conjecture 9.8]). *Let $p \equiv 1, 9 \pmod{40}$ be a prime and hence $p = c^2 + d^2 = x^2 + 10y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv x \equiv 1 \pmod{4}$. Then*

$$5^{\frac{p-1}{8}} \equiv \begin{cases} \pm(-1)^{\frac{d}{4} + \frac{x-1}{4}}\frac{d}{c} \pmod{p} & \text{if } x \equiv \pm d \pmod{5}, \\ \pm(-1)^{\frac{d}{4} + \frac{x-1}{4}} \pmod{p} & \text{if } x \equiv \pm c \pmod{5}. \end{cases}$$

Conjecture 2.5 ([S3, Conjecture 9.9]). *Let $p \equiv 1, 9, 17, 25, 29, 49 \pmod{52}$ be a prime and hence $p = c^2 + d^2 = x^2 + 13y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$ and all the odd parts of d, x, y are congruent to 1 modulo 4.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$13^{\frac{p-1}{8}} \equiv \begin{cases} \mp(-1)^{\frac{d}{4}}\delta(y)\frac{d}{c} \pmod{p} & \text{if } \frac{2c+3d}{x} \equiv \pm 1, \pm 3, \pm 9 \pmod{13}, \\ \pm(-1)^{\frac{d}{4}}\delta(y) \pmod{p} & \text{if } \frac{2c+3d}{x} \equiv \pm 2, \pm 5, \pm 6 \pmod{13}. \end{cases}$$

(ii) *If $p \equiv 5 \pmod{8}$, then*

$$13^{\frac{p-5}{8}} \equiv \begin{cases} \pm\delta(x)\frac{y}{x} \pmod{p} & \text{if } \frac{2c+3d}{x} \equiv \pm 1, \pm 3, \pm 9 \pmod{13}, \\ \pm\delta(x)\frac{dy}{cx} \pmod{p} & \text{if } \frac{2c+3d}{x} \equiv \pm 2, \pm 5, \pm 6 \pmod{13}. \end{cases}$$

Conjecture 2.6 ([S3, Conjecture 9.16]). *Let $p \equiv 1 \pmod{4}$ be a prime and $p = c^2 + d^2 = x^2 + 17y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$ and all the odd parts of d, x, y are numbers of the form $4k + 1$.*

(i) *If $p \equiv 1 \pmod{8}$, then*

$$17^{\frac{p-1}{8}} \equiv \begin{cases} -(-1)^{\frac{d}{4} + \frac{xy}{4}}\frac{d}{c} \pmod{p} & \text{if } 4c + d \equiv \pm 6x, \pm 7x \pmod{17}, \\ (-1)^{\frac{d}{4} + \frac{xy}{4}}\frac{d}{c} \pmod{p} & \text{if } 4c + d \equiv \pm 3x, \pm 5x \pmod{17}, \\ (-1)^{\frac{d}{4} + \frac{xy}{4}} \pmod{p} & \text{if } 4c + d \equiv \pm x, \pm 4x \pmod{17}, \\ -(-1)^{\frac{d}{4} + \frac{xy}{4}} \pmod{p} & \text{if } 4c + d \equiv \pm 2x, \pm 8x \pmod{17}. \end{cases}$$

(ii) If $p \equiv 5 \pmod{8}$, then

$$17^{\frac{p-5}{8}} \equiv \begin{cases} (-1)^x \frac{y}{x} \pmod{p} & \text{if } 4c + d \equiv \pm 6x, \pm 7x \pmod{17}, \\ -(-1)^x \frac{y}{x} \pmod{p} & \text{if } 4c + d \equiv \pm 3x, \pm 5x \pmod{17}, \\ (-1)^x \frac{dy}{cx} \pmod{p} & \text{if } 4c + d \equiv \pm x, \pm 4x \pmod{17}, \\ -(-1)^x \frac{dy}{cx} \pmod{p} & \text{if } 4c + d \equiv \pm 2x, \pm 8x \pmod{17}. \end{cases}$$

Conjecture 2.7 ([S5, Conjecture 4.3]). Let $p \equiv 1 \pmod{4}$ and $q \equiv 3 \pmod{8}$ be primes such that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$ and $q \mid cd$. Suppose $c \equiv 1 \pmod{4}$, $x = 2^\alpha x_0$, $y = 2^\beta y_0$ and $x_0 \equiv y_0 \equiv 1 \pmod{4}$.

(i) If $p \equiv 1 \pmod{8}$, then

$$q^{\frac{p-1}{8}} \equiv \begin{cases} \pm(-1)^{\frac{y}{4}} \pmod{p} & \text{if } x \equiv \pm c \pmod{q}, \\ \mp(-1)^{\frac{q-3}{8} + \frac{y}{4}} \frac{d}{c} \pmod{p} & \text{if } x \equiv \pm d \pmod{q}. \end{cases}$$

(ii) If $p \equiv 5 \pmod{8}$, then

$$q^{\frac{p-5}{8}} \equiv \begin{cases} \pm \frac{y}{x} \pmod{p} & \text{if } x \equiv \pm c \pmod{q}, \\ \mp(-1)^{\frac{q-3}{8}} \frac{dy}{cx} \pmod{p} & \text{if } x \equiv \pm d \pmod{q}. \end{cases}$$

Conjecture 2.8 ([S5, Conjecture 4.4]). Let $p \equiv 1 \pmod{4}$ and $q \equiv 7 \pmod{16}$ be primes such that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$ and $q \mid cd$. Suppose $c \equiv 1 \pmod{4}$, $x = 2^\alpha x_0$, $y = 2^\beta y_0$ and $x_0 \equiv y_0 \equiv 1 \pmod{4}$.

(i) If $p \equiv 1 \pmod{8}$, then

$$q^{\frac{p-1}{8}} \equiv \begin{cases} (-1)^{\frac{y}{4}} \pmod{p} & \text{if } q \mid d, \\ -(-1)^{\frac{y}{4}} \pmod{p} & \text{if } q \mid c. \end{cases}$$

(ii) If $p \equiv 5 \pmod{8}$, then

$$q^{\frac{p-5}{8}} \equiv \begin{cases} \frac{y}{x} \pmod{p} & \text{if } q \mid d, \\ -\frac{y}{x} \pmod{p} & \text{if } q \mid c. \end{cases}$$

Conjecture 2.9 ([S5, Conjecture 4.5]). Let $p \equiv 1 \pmod{4}$ and $q \equiv 15 \pmod{16}$ be primes such that $p = c^2 + d^2 = x^2 + qy^2$ with $c, d, x, y \in \mathbb{Z}$ and $q \mid cd$. Suppose $c \equiv 1 \pmod{4}$, $x = 2^\alpha x_0$, $y = 2^\beta y_0$ and $x_0 \equiv y_0 \equiv 1 \pmod{4}$.

(i) If $p \equiv 1 \pmod{8}$, then $q^{\frac{p-1}{8}} \equiv (-1)^{\frac{y}{4}} \pmod{p}$.

(ii) If $p \equiv 5 \pmod{8}$, then $q^{\frac{p-5}{8}} \equiv \frac{y}{x} \pmod{p}$.

Conjectures 2.7-2.9 have been checked for all primes $p < 100,000$ and $q < 100$.

Conjecture 2.10 ([S3, Conjecture 9.5]). *Let $p \equiv 1 \pmod{4}$ be a prime, $b \in \mathbb{Z}$, $2 \nmid b$, $p \neq b^2 + 4$ and $p = c^2 + d^2 = x^2 + (b^2 + 4)y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$ and all the odd parts of d, x, y are numbers of the form $4k + 1$.*

(i) *If $4 \nmid xy$, then*

$$\begin{aligned} \left(\frac{b + \frac{cx}{dy}}{2}\right)^{\frac{p-1}{4}} &\equiv -\left(\frac{b - \frac{cx}{dy}}{2}\right)^{\frac{p-1}{4}} \\ &\equiv \begin{cases} -(-1)^{\frac{d}{4}} \frac{d}{c} \pmod{p} & \text{if } 2 \parallel x \text{ and } b \equiv 1, 3 \pmod{8}, \\ (-1)^{\frac{d}{4}} \frac{d}{c} \pmod{p} & \text{if } 2 \parallel x \text{ and } b \equiv 5, 7 \pmod{8}, \\ 1 \pmod{p} & \text{if } 2 \parallel y. \end{cases} \end{aligned}$$

(ii) *If $4 \mid xy$, then*

$$\begin{aligned} \left(\frac{b + \frac{cx}{dy}}{2}\right)^{\frac{p-1}{4}} &\equiv \left(\frac{b - \frac{cx}{dy}}{2}\right)^{\frac{p-1}{4}} \\ &\equiv \begin{cases} (-1)^{\frac{d+y}{4}} \pmod{p} & \text{if } 4 \mid y, \\ -(-1)^{\frac{x}{4}} \frac{d}{c} \pmod{p} & \text{if } 4 \mid x \text{ and } b \equiv 1, 3 \pmod{8}, \\ (-1)^{\frac{x}{4}} \frac{d}{c} \pmod{p} & \text{if } 4 \mid x \text{ and } b \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

We remark that Conjectures 2.1-2.10 have been solved by the author under some restricted conditions in [S9].

3. Conjectures on Lucas sequences.

Conjecture 3.1 ([S3, Conjecture 9.4]). *Let $p \equiv 1 \pmod{4}$ be a prime, $b \in \mathbb{Z}$, $2 \nmid b$ and $p = c^2 + d^2 = x^2 + (b^2 + 4)y^2 \neq b^2 + 4$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$ and all the odd parts of d, x, y are numbers of the form $4k + 1$.*

(i) *If $4 \nmid xy$, then*

$$U_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} (-1)^{\frac{d}{4}} \frac{2y}{x} \pmod{p} & \text{if } 2 \parallel x \text{ and } b \equiv 1, 3 \pmod{8}, \\ -(-1)^{\frac{d}{4}} \frac{2y}{x} \pmod{p} & \text{if } 2 \parallel x \text{ and } b \equiv 5, 7 \pmod{8}, \\ \frac{2dy}{cx} \pmod{p} & \text{if } 2 \parallel y. \end{cases}$$

(ii) *If $4 \mid xy$, then*

$$V_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} 2(-1)^{\frac{d+y}{4}} \pmod{p} & \text{if } 4 \mid y, \\ -2(-1)^{\frac{x}{4}} \frac{d}{c} \pmod{p} & \text{if } 4 \mid x \text{ and } b \equiv 1, 3 \pmod{8}, \\ 2(-1)^{\frac{x}{4}} \frac{d}{c} \pmod{p} & \text{if } 4 \mid x \text{ and } b \equiv 5, 7 \pmod{8}. \end{cases}$$

Conjecture 3.1 has been checked for $1 \leq b < 60$ and $p < 20,000$. When $p \equiv 1 \pmod{8}$, $b = 1, 3$ and $4 \mid y$, the conjecture $V_{\frac{p-1}{4}}(b, -1) \equiv 2(-1)^{\frac{d+y}{4}} \pmod{p}$ is equivalent to a conjecture of E. Lehmer. See [L, Conjecture 4].

By (1.3) and (1.4), Conjecture 3.1 is equivalent to Conjecture 2.11. By [S3], Conjectures 2.3 and 2.5 are consequences of Conjecture 3.1.

Conjecture 3.2 ([S3, Conjecture 9.11]). *Let $p \equiv 1 \pmod{4}$ be a prime, $b \in \mathbb{Z}$, $b \equiv 4 \pmod{8}$, $p \neq b^2/4 + 1$ and $p = c^2 + d^2 = x^2 + (1 + b^2/4)y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$ and all the odd parts of d, x, y are numbers of the form $4k + 1$. Then*

$$U_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} (-1)^{\frac{b+4}{8} + \frac{d}{4}} \frac{y}{x} \pmod{p} & \text{if } 2 \parallel x, \\ \frac{dy}{cx} \pmod{p} & \text{if } 2 \parallel y, \\ 0 \pmod{p} & \text{if } 4 \mid xy \end{cases}$$

and

$$V_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} 2(-1)^{\frac{d}{4} + \frac{y}{4}} \pmod{p} & \text{if } 4 \mid y, \\ 2(-1)^{\frac{b-4}{8} + \frac{x}{4}} \frac{d}{c} \pmod{p} & \text{if } 4 \mid x, \\ 0 \pmod{p} & \text{if } 4 \nmid xy. \end{cases}$$

Conjecture 3.2 has been checked for $1 \leq b \leq 100$ and $p < 20,000$.

Conjecture 3.3 ([S3, Conjecture 9.14]). *Let $p \equiv 1 \pmod{4}$ be a prime, $b \in \mathbb{Z}$, $8 \mid b$, $p \neq b^2/4 + 1$ and $p = c^2 + d^2 = x^2 + (1 + b^2/4)y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$ and all the odd parts of d, x, y are numbers of the form $4k + 1$. Then*

$$U_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} 0 \pmod{p} & \text{if } 4 \mid xy, \\ -(-1)^{(\frac{b}{8}-1)y} \frac{dy}{cx} \pmod{p} & \text{if } 4 \nmid xy \end{cases}$$

and

$$V_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} 2(-1)^{\frac{d}{4} + \frac{xy}{4} + \frac{b}{8}y} \pmod{p} & \text{if } 4 \mid xy, \\ 0 \pmod{p} & \text{if } 4 \nmid xy. \end{cases}$$

Conjecture 3.3 has been checked for $1 \leq b < 100$ and $p < 20,000$. By [S3], Conjecture 3.3 implies Conjecture 2.6.

Conjecture 3.4 ([S3, Conjecture 9.17]). *Let $p \equiv 1 \pmod{4}$ be a prime, $b \in \mathbb{Z}$, $b \equiv 2 \pmod{4}$, $p \neq b^2/4 + 1$ and $p = c^2 + d^2 = x^2 + (1 + b^2/4)y^2$ for some $c, d, x, y \in \mathbb{Z}$. Suppose $c \equiv 1 \pmod{4}$, $x = 2^\alpha x_0$, $y = 2^\beta y_0$ and $x_0 \equiv y_0 \equiv 1 \pmod{4}$. Then*

$$U_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} (-1)^{\frac{b-2}{4} + \frac{d}{4}} \frac{y}{x} \pmod{p} & \text{if } 2 \parallel y, \\ 0 \pmod{p} & \text{if } 4 \mid y \end{cases}$$

and

$$V_{\frac{p-1}{4}}(b, -1) \equiv \begin{cases} 0 \pmod{p} & \text{if } 2 \parallel y, \\ 2(-1)^{\frac{d}{4} + \frac{y}{4}} \pmod{p} & \text{if } 4 \mid y. \end{cases}$$

Conjecture 3.4 has been checked for $1 \leq b < 100$ and $p < 20,000$. By [S3], Conjecture 3.4 implies Conjecture 2.4.

Conjecture 3.5 ([S5, Conjecture 4.1]). *Let p be an odd prime and $k \in \mathbb{N}$ with $2 \nmid k$. Suppose $p = x^2 + (k^2 + 1)y^2$ for some $x, y \in \mathbb{Z}$. Then*

$$V_{\frac{p+1}{4}}(2k, -1) \equiv \begin{cases} (-1)^{\frac{(\frac{p-1}{2}y)^2 - 1}{8}} (-2)^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 5, 7 \pmod{8}, \\ -(-1)^{\frac{(\frac{p-1}{2}y)^2 - 1}{8}} (-2)^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 1, 3 \pmod{8}. \end{cases}$$

Conjecture 3.5 has been checked for all $k < 60$ and $p < 20,000$. The case $k = 1$ has been solved by the author in [S3].

Conjecture 3.6 ([S5, Conjecture 4.2]). *Let p be an odd prime and $k \in \mathbb{N}$ with $2 \nmid k$. Suppose $2p = x^2 + (k^2 + 4)y^2$ for some $x, y \in \mathbb{Z}$.*

(i) *If $k \equiv 1, 3 \pmod{8}$, then*

$$V_{\frac{p+1}{4}}(k, -1) \equiv \begin{cases} (-1)^{\frac{(\frac{p-1}{2}y)^2 - 1}{8}} (-2)^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 1, 11 \pmod{16}, \\ -(-1)^{\frac{(\frac{p-1}{2}y)^2 - 1}{8}} (-2)^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 3, 9 \pmod{16}. \end{cases}$$

(ii) *If $k \equiv 5, 7 \pmod{8}$, then*

$$V_{\frac{p+1}{4}}(k, -1) \equiv \begin{cases} (-1)^{\frac{(\frac{p-1}{2}y)^2 - 1}{8}} 2^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 5, 15 \pmod{16}, \\ -(-1)^{\frac{(\frac{p-1}{2}y)^2 - 1}{8}} 2^{\frac{p+1}{4}} \pmod{p} & \text{if } k \equiv 7, 13 \pmod{16}. \end{cases}$$

Conjecture 3.6 has been checked for all $k < 60$ and $p < 20,000$. The case $k = 1$ was posed by the author in [S1] and solved by C.N. Beli in [B].

Conjecture 3.7 ([S6, Conjecture 5.1]). *Let $p > 7$ be a prime such that $p \equiv 1, 2, 4 \pmod{7}$ and so $p = C^2 + 7D^2$ with $C, D \in \mathbb{Z}$.*

(i) *If $p \equiv 1 \pmod{4}$ and $C \equiv 1 \pmod{4}$, then*

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 U_k(16, 1) &\equiv 0 \pmod{p^2}, \\ \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 V_k(16, 1) &\equiv (-1)^{\frac{p-1}{4}} (4C - \frac{p}{C}) \pmod{p^2}. \end{aligned}$$

(ii) *If $p \equiv 3 \pmod{4}$ and $D \equiv 1 \pmod{4}$, then*

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 U_k(16, 1) &\equiv (-1)^{\frac{p-3}{4}} \left(\frac{16}{3}D - \frac{4p}{21D} \right) \pmod{p^2}, \\ \sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 V_k(16, 1) &\equiv (-1)^{\frac{p+1}{4}} \left(84D - \frac{3p}{D} \right) \pmod{p^2}. \end{aligned}$$

In [S6], the author proved the congruences modulo p .

Conjecture 3.8 ([S6, Conjecture 5.2]). *Let $p > 3$ be a prime such that $p \equiv 1, 3, 4, 9, 10, 12 \pmod{13}$. Then*

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^2 U_k(11, 1) \equiv 0 \pmod{p}.$$

4. Conjectures on supercongruences.

In 2003, Rodriguez-Villegas[RV] posed many conjectures on supercongruences. In particular, he conjectured that for any prime $p > 3$,

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{(6k)!}{1728^k (3k)! k!^3} &\equiv \begin{cases} \left(\frac{p}{3}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2 + y^2 \text{ with } 2 \nmid x, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4} \end{cases} \\ \sum_{k=0}^{p-1} \frac{(4k)!}{256^k k!^4} &\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 3 \pmod{8} \text{ and so } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8} \end{cases} \end{aligned}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } 3 \mid p-1 \text{ and } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } 3 \nmid p-1. \end{cases}$$

The three conjectures have been solved by E. Mortenson[M] and Zhi-Wei Sun[Su2]. Recently my twin brother Zhi-Wei Sun made a lot of conjectures on supercongruences. Inspired by his work in [Su1], the author made the following conjectures.

Conjecture 4.1 ([S4, 2.1]). *Let $p \equiv 1 \pmod{4}$ be a prime and so $p = x^2 + y^2$ with $2 \nmid x$. Then*

$$\sum_{k=0}^{p-1} \frac{(4k)!}{648^k k!^4} \equiv 4x^2 - 2p \pmod{p^2}.$$

In [S6], the author proved the congruence modulo p . When $p \equiv 3 \pmod{4}$, the author proved $\sum_{k=0}^{p-1} \frac{(4k)!}{648^k k!^4} \equiv 0 \pmod{p^2}$ in [S8].

Conjecture 4.2 ([S4, 2.2]). *Let $p \equiv 1 \pmod{3}$ be a prime and so $p = x^2 + 3y^2$. Then*

$$\sum_{k=0}^{p-1} \frac{(4k)!}{(-144)^k k!^4} \equiv 4x^2 - 2p \pmod{p^2}.$$

In [S6], the author proved the congruence modulo p . When $p \equiv 5 \pmod{6}$, the author proved $\sum_{k=0}^{p-1} \frac{(4k)!}{(-144)^k k!^4} \equiv 0 \pmod{p^2}$ in [S8].

Conjecture 4.3 ([S4, 2.3]). *Let $p \equiv 1, 2, 4 \pmod{7}$ be an odd prime and so $p = x^2 + 7y^2$. Then*

$$\sum_{k=0}^{p-1} \frac{(4k)!}{(-3969)^k k!^4} \equiv 4x^2 - 2p \pmod{p^2}.$$

In [S6], the author proved the congruence modulo p . When $p \equiv 3, 5, 6 \pmod{7}$, the author proved $\sum_{k=0}^{p-1} \frac{(4k)!}{(-3969)^k k!^4} \equiv 0 \pmod{p^2}$ in [S8].

Conjecture 4.4 ([S4, 2.4]). *Let $p \equiv 1 \pmod{4}$ be a prime and so $p = x^2 + y^2$ with $2 \nmid x$. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{66^{3k} (3k)! k!^3} \equiv \left(\frac{p}{33}\right) (4x^2 - 2p) \pmod{p^2}.$$

In [S7], the author proved the congruence modulo p . When $p \equiv 3 \pmod{4}$, the author proved $\sum_{k=0}^{p-1} \frac{(6k)!}{66^{3k} (3k)! k!^3} \equiv 0 \pmod{p^2}$ in [S8].

Conjecture 4.5 ([S4, 2.5]). *Let $p \equiv 1, 3 \pmod{8}$ be a prime and so $p = x^2 + 2y^2$. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{20^{3k} (3k)! k!^3} \equiv \left(\frac{-5}{p}\right) (4x^2 - 2p) \pmod{p^2}.$$

In [S7], the author proved the congruence modulo p . When $p \equiv 5, 7 \pmod{8}$, the author proved $\sum_{k=0}^{p-1} \frac{(6k)!}{20^{3k} (3k)! k!^3} \equiv 0 \pmod{p^2}$ in [S8].

Conjecture 4.6 ([S4, 2.6]). *Let $p \equiv 1 \pmod{3}$ be a prime and so $p = x^2 + 3y^2$. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{54000^k (3k)! k!^3} \equiv \left(\frac{p}{5}\right) (4x^2 - 2p) \pmod{p^2}.$$

In [S7], the author proved the congruence modulo p . When $p \equiv 5 \pmod{6}$, the author proved $\sum_{k=0}^{p-1} \frac{(6k)!}{54000^{3k} (3k)! k!^3} \equiv 0 \pmod{p^2}$ in [S8].

Conjecture 4.7 ([S4, 2.7]). *Let $p > 5$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{(6k)!}{(-12288000)^k (3k)! k!^3} \\ & \equiv \begin{cases} \left(\frac{10}{p}\right) (L^2 - 2p) \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and so } 4p = L^2 + 27M^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Conjecture 4.8 ([S4, 2.8]). *Let $p \equiv 1, 2, 4 \pmod{7}$ be an odd prime and so $p = x^2 + 7y^2$. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{(-15)^{3k}(3k)!k!^3} \equiv \left(\frac{p}{15}\right)(4x^2 - 2p) \pmod{p^2}.$$

In [S7], the author proved the congruence modulo p . When $p \equiv 3, 5, 6 \pmod{7}$, the author proved $\sum_{k=0}^{p-1} \frac{(6k)!}{(-15)^{3k}(3k)!k!^3} \equiv 0 \pmod{p^2}$ in [S8].

Conjecture 4.9 ([S4, 2.9]). *Let $p \equiv 1, 2, 4 \pmod{7}$ be an odd prime and so $p = x^2 + 7y^2$. Then*

$$\sum_{k=0}^{p-1} \frac{(6k)!}{255^{3k}(3k)!k!^3} \equiv \left(\frac{p}{255}\right)(4x^2 - 2p) \pmod{p^2}.$$

In [S7], the author proved the congruence modulo p . When $p \equiv 3, 5, 6 \pmod{7}$, the author proved $\sum_{k=0}^{p-1} \frac{(6k)!}{255^{3k}(3k)!k!^3} \equiv 0 \pmod{p^2}$ in [S8].

Conjecture 4.10 ([S4, 2.10]). *Let $p > 3$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{1458^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \text{ and so } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Conjecture 4.11 ([S4, 2.11]). *Let $p > 5$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{15^{3k}} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and so } p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ and so } p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 7, 11, 13, 14 \pmod{15}. \end{cases} \end{aligned}$$

Conjecture 4.12 ([S4, 2.12]). *Let $p > 3$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{(-8640)^k} \\ & \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } 3 \mid p-1, p = x^2 + 3y^2 \text{ and } 10^{\frac{p-1}{3}} \equiv 1 \pmod{p}, \\ p - 2x^2 \pm 6xy \pmod{p^2} & \text{if } 3 \mid p-1, p = x^2 + 3y^2 \text{ and } 10^{\frac{p-1}{3}} \equiv \frac{1}{2}(-1 \mp \frac{x}{y}) \pmod{p}, \\ 0 \pmod{p^2} & \text{if } 3 \mid p-2. \end{cases} \end{aligned}$$

If p is a prime of the form $3k + 1$ and so $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$, by [S2, Corollary 4.4] we have $10^{\frac{p-1}{3}} \equiv 1 \pmod{p}$ if and only if $5 \mid xy$.

Conjecture 4.13 ([S6, Conjecture 5.3]). *Let $p \equiv 1 \pmod{12}$ be a prime and $p = a^2 + 4b^2 = A^2 + 3B^2$ with $a \equiv A \equiv 1 \pmod{4}$. Then*

$$\sum_{k=0}^{(p-1)/6} \frac{\binom{6k}{3k}^2}{(-16)^{3k}} \equiv (-1)^{\frac{p-1}{4}} \frac{1}{3} \left(2a + 4A - \frac{p}{2a} - \frac{p}{A} \right) \pmod{p^2}.$$

In [S6], the author proved the congruence modulo p .

For an integer m and odd prime p with $p \nmid m$ let

$$S_p(m) = \sum_{n=0}^{p-1} \sum_{k=0}^n \binom{n}{k}^4 m^k.$$

Recently Zhi-Wei Sun [Su5] investigated congruences $S_p(m) \pmod{p^2}$ and revealed the connections with binary quadratic forms and series for $\frac{1}{\pi}$. Now we introduce the sum

$$Z_p(m) = \sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{m^n} \sum_{k=0}^n \binom{n}{k}^3.$$

Then we have the following conjectures concerning $Z_p(m)$ modulo p^2 .

Conjecture 4.14. *Let p be an odd prime. Then*

$$Z_p(-16) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{12} \text{ with } 6 \mid y, \\ 2p - 4x^2 \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{12} \text{ with } 6 \mid x - 3, \\ 4\left(\frac{xy}{3}\right)xy \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 5 \pmod{12}, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Conjecture 4.15. *Let p be an odd prime. Then*

$$Z_p(96) \equiv \begin{cases} \left(\frac{p}{3}\right)(4x^2 - 2p) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1, 3 \pmod{8}, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Conjecture 4.16. *Let $p > 5$ be a prime. Then*

$$Z_p(50) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Conjecture 4.17. *Let $p > 5$ be a prime. Then*

$$Z_p(16) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \text{ and so } p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \text{ and so } 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

Conjecture 4.18. *Let $p > 3$ be a prime. Then*

$$Z_p(32) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and so } p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ and so } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

Conjecture 4.19. *Let $p > 7$ be a prime. Then*

$$Z_p(5) \equiv Z_p(-49) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 15y^2 \equiv 1, 19 \pmod{30}, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + 5y^2 \equiv 17, 23 \pmod{30}, \\ 0 \pmod{p^2} & \text{if } p \equiv 7, 11, 13, 29 \pmod{30}. \end{cases}$$

Conjecture 4.20. *Let $b \in \{7, 11, 19, 31, 59\}$ and*

$$f(b) = \begin{cases} -112 & \text{if } b = 7, \\ -400 & \text{if } b = 11, \\ -2704 & \text{if } b = 19, \\ -24304 & \text{if } b = 31, \\ -1123600 & \text{if } b = 59. \end{cases}$$

If p is a prime with $p \neq 2, 3, b$ and $p \nmid f(b)$, then

$$Z_p(f(b)) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3by^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + by^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } 2p = x^2 + 3by^2, \\ 2p - 6x^2 \pmod{p^2} & \text{if } 2p = 3x^2 + by^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-3b}{p}\right) = -1. \end{cases}$$

Conjecture 4.21. *Let $b \in \{5, 7, 13, 17\}$ and*

$$f(b) = \begin{cases} 320 & \text{if } b = 5, \\ 896 & \text{if } b = 7, \\ 10400 & \text{if } b = 13, \\ 39200 & \text{if } b = 17. \end{cases}$$

If p is a prime with $p \neq 2, 3, b$ and $p \nmid f(b)$, then

$$Z_p(f(b)) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 6by^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p = 2x^2 + 3by^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + 2by^2, \\ 2p - 24x^2 \pmod{p^2} & \text{if } p = 6x^2 + by^2, \\ 0 \pmod{p^2} & \text{if } \left(\frac{-6b}{p}\right) = -1. \end{cases}$$

Conjecture 4.22. *Let $p > 3$ be a prime. Then*

$$\begin{aligned}
\sum_{n=0}^{p-1} \frac{9n+4}{5^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 4\left(\frac{p}{5}\right)p \pmod{p^2} \quad \text{for } p > 5, \\
\sum_{n=0}^{p-1} \frac{5n+2}{16^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 2p \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{9n+2}{50^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 2\left(\frac{-1}{p}\right)p \pmod{p^2} \quad \text{for } p \neq 5, \\
\sum_{n=0}^{p-1} \frac{5n+1}{96^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv \left(\frac{-2}{p}\right)p \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{6n+1}{320^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv \left(\frac{p}{15}\right)p \pmod{p^2} \quad \text{for } p \neq 5, \\
\sum_{n=0}^{p-1} \frac{90n+13}{896^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 13\left(\frac{p}{7}\right)p \pmod{p^2} \quad \text{for } p \neq 7, \\
\sum_{n=0}^{p-1} \frac{102n+11}{10400^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 11\left(\frac{p}{39}\right)p \pmod{p^2} \quad \text{for } p \neq 5, 13
\end{aligned}$$

Conjecture 4.23. *Let $p > 3$ be a prime. Then*

$$\begin{aligned}
\sum_{n=0}^{p-1} \frac{3n+1}{(-16)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv \left(\frac{-1}{p}\right)p \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{15n+4}{(-49)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 4\left(\frac{p}{3}\right)p \pmod{p^2} \quad \text{for } p \neq 5, 7, \\
\sum_{n=0}^{p-1} \frac{9n+2}{(-112)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 2\left(\frac{p}{7}\right)p \pmod{p^2} \quad \text{for } p \neq 7, \\
\sum_{n=0}^{p-1} \frac{99n+17}{(-400)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 17\left(\frac{-1}{p}\right)p \pmod{p^2}, \\
\sum_{n=0}^{p-1} \frac{855n+109}{(-2704)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 109p\left(\frac{-1}{p}\right) \pmod{p^2} \quad \text{for } p \neq 13, \\
\sum_{n=0}^{p-1} \frac{585n+58}{(-24304)^n} \binom{2n}{n} \sum_{k=0}^n \binom{n}{k}^3 &\equiv 58p\left(\frac{-31}{p}\right) \pmod{p^2} \quad \text{for } p \neq 7, 31.
\end{aligned}$$

Conjecture 4.24. *Let $p > 5$ be a prime. Then*

$$\begin{aligned}
\sum_{k=0}^{p-1} \frac{63k+8}{(-15)^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 8p \left(\frac{-15}{p} \right) \pmod{p^2}, \\
\sum_{k=0}^{p-1} \frac{133k+8}{255^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 8p \left(\frac{-255}{p} \right) \pmod{p^2} \quad \text{for } p \neq 17, \\
\sum_{k=0}^{p-1} \frac{28k+3}{20^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 3p \left(\frac{-5}{p} \right) \pmod{p^2}, \\
\sum_{k=0}^{p-1} \frac{63k+5}{66^{3k}} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 5p \left(\frac{-33}{p} \right) \pmod{p^2} \quad \text{for } p \neq 11, \\
\sum_{k=0}^{p-1} \frac{11k+1}{54000^k} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv p \left(\frac{-15}{p} \right) \pmod{p^2}, \\
\sum_{k=0}^{p-1} \frac{506k+31}{(-12288000)^k} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} &\equiv 31p \left(\frac{-30}{p} \right) \pmod{p^2}.
\end{aligned}$$

Conjecture 4.25. *Let $p > 5$ be a prime. Then*

$$\begin{aligned}
\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} (-4)^k &\equiv \left(\frac{5}{p} \right) 5^{\frac{1-(\frac{p}{3})}{2}} \sum_{k=0}^{p-1} \binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k} (-4)^k \\
&\equiv \begin{cases} \left(\frac{x}{3} \right) (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 15y^2, \\ -\left(\frac{x}{3} \right) (10x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = 5x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{30} \end{cases}
\end{aligned}$$

and so

$$2x \left(\frac{x}{3} \right) \equiv \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} (-4)^k \pmod{p} \quad \text{for } p = 5x^2 + 3y^2.$$

Conjecture 4.26. *Let $p > 3$ be a prime. Then*

$$\begin{aligned}
\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k}}{2^k} &\equiv \left(\frac{2}{p} \right) 2^{\frac{(\frac{p}{3})-1}{2}} \sum_{k=0}^{p-1} \frac{\binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k}}{2^k} \\
&\equiv \begin{cases} \left(\frac{x}{3} \right) (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 6y^2 \equiv 1, 7 \pmod{24}, \\ \left(\frac{x}{3} \right) (2x - \frac{p}{4x}) \pmod{p^2} & \text{if } p = 2x^2 + 3y^2 \equiv 5, 11 \pmod{24}, \\ 0 \pmod{p} & \text{if } p \equiv 13, 19 \pmod{24} \end{cases}
\end{aligned}$$

and so

$$x \left(\frac{x}{3} \right) \equiv -\frac{1}{4} \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} \frac{1}{2^k} \pmod{p} \quad \text{for } p = 2x^2 + 3y^2.$$

Conjecture 4.27. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k}}{(-16)^k} &\equiv \left(\frac{17}{p}\right) \left(\frac{17}{16}\right)^{\frac{1-(\frac{p}{3})}{2}} \sum_{k=0}^{p-1} \frac{\binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k}}{(-16)^k} \\ &\equiv \begin{cases} -(\frac{x}{3})(x - \frac{p}{x}) \pmod{p^2} & \text{if } 4p = x^2 + 51y^2, \\ \frac{1}{4}(\frac{x}{3})(17x - \frac{p}{x}) \pmod{p^2} & \text{if } 4p = 17x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } (\frac{p}{3}) = -(\frac{p}{17}) = 1 \end{cases} \end{aligned}$$

and so

$$x\left(\frac{x}{3}\right) \equiv -\frac{1}{4} \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} \frac{1}{(-16)^k} \pmod{p} \quad \text{for } 4p = 17x^2 + 3y^2.$$

Conjecture 4.28. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k}}{(-1024)^k} &\equiv \left(\frac{41}{p}\right) \left(\frac{1025}{1024}\right)^{\frac{1-(\frac{p}{3})}{2}} \sum_{k=0}^{p-1} \frac{\binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k}}{(-1024)^k} \\ &\equiv \begin{cases} -(\frac{x}{3})(x - \frac{p}{x}) \pmod{p^2} & \text{if } 4p = x^2 + 123y^2, \\ \frac{5}{32}(\frac{x}{3})(41x - \frac{p}{x}) \pmod{p^2} & \text{if } 4p = 41x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } (\frac{p}{3}) = -(\frac{p}{41}) = 1 \end{cases} \end{aligned}$$

and so

$$x\left(\frac{x}{3}\right) \equiv -\frac{5}{32} \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} \frac{1}{(-1024)^k} \pmod{p} \quad \text{for } 4p = 41x^2 + 3y^2.$$

Conjecture 4.29. *Let $p > 3$ be a prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k}}{(-250000)^k} &\equiv \left(\frac{89}{p}\right) \left(\frac{250001}{250000}\right)^{\frac{1-(\frac{p}{3})}{2}} \sum_{k=0}^{p-1} \frac{\binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k}}{(-250000)^k} \\ &\equiv \begin{cases} -(\frac{x}{3})(x - \frac{p}{x}) \pmod{p^2} & \text{if } 4p = x^2 + 267y^2, \\ \frac{53}{500}(\frac{x}{3})(89x - \frac{p}{x}) \pmod{p^2} & \text{if } 4p = 89x^2 + 3y^2, \\ 0 \pmod{p} & \text{if } (\frac{p}{3}) = -(\frac{p}{89}) = 1 \end{cases} \end{aligned}$$

and so

$$x\left(\frac{x}{3}\right) \equiv -\frac{53}{500} \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} \frac{1}{(-250000)^k} \pmod{p} \quad \text{for } 4p = 89x^2 + 3y^2.$$

Conjecture 4.30. *Let $p > 5$ be a prime. Then*

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} \frac{1}{(-80)^k} \\
& \equiv \left(\frac{5}{p}\right) \sum_{k=0}^{p-1} \binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k} \frac{1}{(-80)^k} \\
& \equiv \begin{cases} x - \frac{p}{x} \pmod{p^2} & \text{if } p \equiv 1, 19 \pmod{30} \text{ and so } 4p = x^2 + 75y^2 \text{ with } 3 \mid x - 2, \\ 5x - \frac{p}{5x} \pmod{p^2} & \text{if } p \equiv 7, 13 \pmod{30} \text{ and so } 4p = 25x^2 + 3y^2 \text{ with } 3 \mid x - 1, \\ 0 \pmod{p} & \text{if } p \equiv 17, 23 \pmod{30} \end{cases}
\end{aligned}$$

Conjecture 4.31. *Let $p > 11$ be a prime such that $(\frac{p}{11}) = 1$ and so $4p = x^2 + 11y^2$. Then*

$$\begin{aligned}
\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} \left(\frac{27}{16}\right)^k & \equiv \left(-\frac{11}{16}\right)^{\frac{1-(\frac{p}{3})}{2}} \sum_{k=0}^{p-1} \binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k} \left(\frac{27}{16}\right)^k \\
& \equiv \begin{cases} -\left(\frac{-11+x/y}{p}\right)\left(\frac{x}{11}\right)\left(x - \frac{p}{x}\right) \pmod{p^2} & \text{if } 3 \mid p - 1, \\ -\frac{1}{4}\left(\frac{-11+(\frac{x}{11})x/y}{p}\right)(11y - \frac{p}{y}) \pmod{p^2} & \text{if } 3 \nmid p - 1 \end{cases}
\end{aligned}$$

and so

$$\begin{aligned}
& y\left(\frac{-11 + (\frac{x}{11})x/y}{p}\right) \\
& \equiv \frac{1}{4} \sum_{k=0}^{(p-5)/6} \binom{\frac{p-2}{3}}{k} \binom{\frac{p-5}{6}}{k} \left(\frac{27}{16}\right)^k \pmod{p} \quad \text{for } 4p = x^2 + 11y^2 \equiv 2 \pmod{3}.
\end{aligned}$$

Conjecture 4.32. *Let $p \equiv 1 \pmod{3}$ be a prime and so $4p = L^2 + 27M^2$ with $3 \mid L - 2$. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} \left(-\frac{9}{16}\right)^k \equiv \sum_{k=0}^{p-1} \binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k} \left(-\frac{9}{16}\right)^k \equiv L - \frac{p}{L} \pmod{p^2}.$$

Conjecture 4.33. *Let $p \equiv 1, 3 \pmod{8}$ be a prime and so $p = c^2 + 2d^2$ with $4 \mid c - 1$. Then*

$$\begin{aligned}
& \sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k} \binom{-\frac{1}{6}}{k} \left(\frac{27}{2}\right)^k \\
& \equiv \left(-\frac{25}{2}\right)^{\frac{1-(\frac{p}{3})}{2}} \sum_{k=0}^{p-1} \binom{-\frac{2}{3}}{k} \binom{-\frac{5}{6}}{k} \left(\frac{27}{2}\right)^k \\
& \equiv \begin{cases} (-1)^{[\frac{p}{8}]} \left(\frac{-2-c/d}{p}\right) \left(2c - \frac{p}{2c}\right) \pmod{p^2} & \text{if } p \equiv 1, 19 \pmod{24}, \\ (-1)^{[\frac{p}{8}]} \left(\frac{2+c/d}{p}\right) \left(10d - \frac{5p}{4d}\right) \pmod{p^2} & \text{if } p \equiv 11, 17 \pmod{24}. \end{cases}
\end{aligned}$$

Conjecture 4.34. Let p be a prime such that $p \equiv 1 \pmod{3}$, $4p = L^2 + 27M^2$ ($L, M \in \mathbb{Z}$) and $L \equiv 2 \pmod{3}$. Then

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{3}}{k}^2 9^k \equiv L - \frac{p}{L} \pmod{p^2}$$

Conjecture 4.35. Let p be a prime of the form $4k+1$ and so $p = x^2 + y^2$ with $4 \mid x-1$. Then

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 (-8)^k \equiv (-1)^{\frac{p-1}{4}} (2x - \frac{p}{2x}) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k}^2}{(-8)^k} \equiv \begin{cases} (-1)^{\frac{y}{4}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p \equiv 1 \pmod{8}, \\ (-1)^{\frac{y-2}{4}} (2y - \frac{p}{2y}) \pmod{p^2} & \text{if } p \equiv 5 \pmod{8}. \end{cases}$$

Conjecture 4.36. Let $p \equiv 1 \pmod{3}$ be a prime and so $p = A^2 + 3B^2$. Then

$$\begin{aligned} \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 4^k &\equiv \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k} (-8)^k \\ &\equiv \begin{cases} (-1)^{\frac{p-1}{4} + \frac{A-1}{2}} (2A - \frac{p}{2A}) \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ (-1)^{\frac{p+1}{4} + \frac{B-1}{2}} (6B - \frac{p}{2B}) \pmod{p^2} & \text{if } p \equiv 7 \pmod{12}. \end{cases} \end{aligned}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k}^2}{4^k} \equiv \begin{cases} (-1)^{\frac{A-1}{2}} (2A - \frac{p}{2A}) \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ (-1)^{\frac{B-1}{2}} (3B - \frac{p}{4B}) \pmod{p^2} & \text{if } p \equiv 7 \pmod{12}. \end{cases}$$

Conjecture 4.37. Let $p > 2$ be a prime such that $p \equiv 1, 2, 4 \pmod{7}$ and so $p = x^2 + 7y^2$. Then

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k}^2 64^k \equiv \begin{cases} (\frac{2}{p})(-1)^{\frac{x-1}{2}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ (\frac{2}{p})(-1)^{\frac{y-1}{2}} (42y - \frac{3p}{2y}) \pmod{p^2} & \text{if } p \equiv 3 \pmod{4} \end{cases}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k}^2}{64^k} \equiv \begin{cases} (-1)^{\frac{x-1}{2}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p \equiv 1 \pmod{4}, \\ \frac{3}{4}(-1)^{\frac{y-1}{2}} (7y - \frac{p}{4y}) \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Conjecture 4.38. *Let p be a prime such that $p \equiv 5, 7 \pmod{8}$. Then*

$$\sum_{k=0}^{p-1} (-1)^k \binom{-\frac{1}{4}}{k}^2 \equiv \begin{cases} (-1)^{\frac{x+1}{2}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 1 \pmod{8}, \\ (-1)^{\frac{y-1}{2}} (4y - \frac{p}{2y}) \pmod{p^2} & \text{if } p = x^2 + 2y^2 \equiv 3 \pmod{8}, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

For the results related to Conjectures 4.25-4.38, see [S10].

Conjecture 4.39. *Let p be an odd prime.*

(i) *If $p \equiv 1 \pmod{4}$ and so $p = x^2 + y^2$ with $2 \nmid x$, then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{4^k} &\equiv \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{6}}{k} 2^k \\ &\equiv \begin{cases} (-1)^{\frac{p-1}{4} + \frac{x+1}{2}} (2x - \frac{p}{2x}) \pmod{p^2} & \text{if } 12 \mid p-1, \\ 2y - \frac{p}{2y} \pmod{p^2} & \text{if } 12 \mid p-5. \end{cases} \end{aligned}$$

(ii) *If $p \equiv 3 \pmod{4}$, then*

$$\sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{4^k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{6}}{k} 2^k \equiv 0 \pmod{p}.$$

Conjecture 4.40. *Let p be an odd prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{(-3)^k} &\equiv (-1)^{\frac{p-1}{4}} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{81^k} \\ &\equiv \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv 1 \pmod{4} \text{ and } 2 \nmid x, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Conjecture 4.41. *Let p be an odd prime. Then*

$$\begin{aligned} \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k}}{(-80)^k} \\ \equiv \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv \pm 1 \pmod{5} \text{ and } 2 \nmid x, \\ 2y - \frac{p}{2y} \pmod{p^2} & \text{if } p = x^2 + y^2 \equiv \pm 2 \pmod{5} \text{ and } 2 \nmid x, \\ 0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}. \end{cases} \end{aligned}$$

Conjecture 4.42. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{4}}{k} \binom{-\frac{1}{2}}{k} 2^k \\ &= \begin{cases} 2x - \frac{p}{2x} \pmod{p^2} & \text{if } p = x^2 + 2y^2 \text{ with } x \equiv 1 \pmod{4}, \\ 0 \pmod{p} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases} \end{aligned}$$

Conjecture 4.43. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k} (-3)^k \equiv \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k}}{(-27)^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{2}{3}}{k}}{(-4)^k} \\ & \equiv \left(\frac{p}{5}\right) \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k}}{5^k} \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k} 2^k \\ & \equiv \begin{cases} 2A - \frac{p}{2A} \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ with } 3 \mid A - 1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Conjecture 4.44. *Let $p > 5$ be a prime. Then*

$$\begin{aligned} & \sum_{k=0}^{p-1} \frac{\binom{-\frac{1}{2}}{k} \binom{-\frac{1}{3}}{k}}{(-4)^k} \\ &= \begin{cases} \left(\frac{p}{5}\right) (2A - \frac{p}{2A}) \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ with } 5 \mid AB \text{ and } 3 \mid A - 1, \\ \left(\frac{p}{5}\right) (A + 3B - \frac{p}{A+3B}) \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3} \text{ with } A/B \equiv -1, -2 \pmod{5} \text{ and } 3 \mid A - 1, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases} \end{aligned}$$

Conjecture 4.45. *Let $p > 5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \binom{-\frac{1}{2}}{k} \binom{-\frac{1}{6}}{k} \left(-\frac{3}{125}\right)^k \equiv \begin{cases} 2A - \frac{p}{2A} \pmod{p^2} & \text{if } p = A^2 + 3B^2 \equiv 1 \pmod{3}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Conjecture 4.46. *Let $p > 3$ be a prime. Then*

(i) *If $p \equiv 1 \pmod{4}$, then*

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{64^k} \equiv \left(\frac{3}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{12^{3k}} \equiv \left(\frac{33}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{66^{3k}} \pmod{p^3}.$$

(ii) If $p \equiv 1, 2, 4 \pmod{7}$, then

$$\sum_{k=0}^{(p-1)/2} \binom{2k}{k}^3 \equiv \left(\frac{-15}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{(-15)^{3k}} \equiv \left(\frac{-255}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{255^{3k}} \pmod{p^3}.$$

(iii) If $p \equiv 1, 3 \pmod{8}$, then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{5}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{20^{3k}} \pmod{p^3}.$$

(iv) If $p \equiv 1 \pmod{3}$, then

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{256^k} \equiv \left(\frac{-5}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} \binom{3k}{k} \binom{6k}{3k}}{54000^k} \pmod{p^3}.$$

Let $p > 3$ be a prime, and $m \in \mathbb{Z}_p$ with $m \not\equiv 0, 16, 64 \pmod{p}$. From [S6] and [S7] we deduce that

$$\begin{aligned} \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^3}{m^k} &\equiv \left(\frac{m(m-64)}{p}\right) P_{\lfloor \frac{p}{4} \rfloor} \left(\frac{m+64}{m-64}\right)^2 \\ &\equiv \left(\frac{m(m-64)}{p}\right) \left(\sum_{x=0}^{p-1} \left(\frac{x^3 - \frac{3}{2}(3 \cdot \frac{m+64}{m-64} + 5)x + 9 \cdot \frac{m+64}{m-64} + 7}{p}\right)^2\right) \\ &\equiv \left(\frac{m(m-16)}{p}\right) \sum_{k=0}^{\lfloor p/6 \rfloor} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{m}{(m-16)^3}\right)^k \\ &\equiv \left(\frac{m(m-16)}{p}\right) \sum_{k=0}^{p-1} \binom{2k}{k} \binom{3k}{k} \binom{6k}{3k} \left(\frac{m}{(m-16)^3}\right)^k \pmod{p}. \end{aligned}$$

REFERENCES

- [B] C.N. Beli, *Two conjectures by Zhi-Hong Sun*, Acta Arith. **137** (2009), 99-131.
- [H] R.H. Hudson, *Diophantine determinations of $3^{(p-1)/8}$ and $5^{(p-1)/4}$* , Pacific J. Math. **111** (1984), 49-55.
- [HW] R.H. Hudson and K.S. Williams, *Some new residuacity criteria*, Pacific J. Math. **91** (1980), 135-143.
- [L] E. Lehmer, *On the quartic character of quadratic units*, J. Reine Angew. Math. **268/269** (1974), 294-301.
- [M] E. Mortenson, *Supercongruences for truncated ${}_nF_n$ hypergeometric series with applications to certain weight three newforms*, Proc. Amer. Math. Soc. **133**(2005), 321-330..
- [RV] F. Rodriguez-Villegas, *Hypergeometric families of Calabi-Yau manifolds*, in: *Calabi-Yau Varieties and Mirror Symmetry (Toronto, ON, 2001)*, pp.223-231, Fields Inst. Commun., 38, Amer. Math. Soc., Providence, RI, 2003..

- [S1] Z.H. Sun, *Values of Lucas sequences modulo primes*, Rocky Mountain J. Math. **33** (2003), 1123-1145.
- [S2] Z.H. Sun, *Cubic residues and binary quadratic forms*, J. Number Theory **124** (2007), 62-104.
- [S3] Z.H. Sun, *Quartic, octic residues and Lucas sequences*, J. Number Theory **129** (2009), 499-550.
- [S4] Z.H. Sun, *Congruences concerning Legendre polynomials*, Proc. Amer. Math. Soc. **139** (2011), 1915-1929.
- [S5] Z.H. Sun, *Congruences for $(A + \sqrt{A^2 + mB^2})^{\frac{p-1}{2}}$ and $(b + \sqrt{a^2 + b^2})^{\frac{p-1}{4}} \pmod{p}$* , Acta Arith. **149** (2011), 275-296.
- [S6] Z.H. Sun, *Congruences concerning Legendre polynomials II*, preprint, arXiv:1012.3898. <http://arxiv.org/abs/1012.3898>.
- [S7] Z.H. Sun, *Congruences concerning Legendre polynomials III*, preprint, arXiv:1012.4234. <http://arxiv.org/abs/1012.4234>.
- [S8] Z.H. Sun, *Some supercongruences modulo p^2* , preprint, arXiv:1101.1050. <http://arxiv.org/abs/1101.1050>.
- [S9] Z.H. Sun, *Quartic, octic residues and binary quadratic forms*, preprint, arXiv:1108.3027. <http://arxiv.org/abs/1108.3027>.
- [S10] Z.H. Sun, *Jacobsthal sums, Legendre polynomials and binary quadratic forms*, preprint, arXiv:1202.1237. <http://arxiv.org/abs/1202.1237>.
- [Su1] Z.W. Sun, *On congruences related to central binomial coefficients*, preprint, arXiv:0911.5615. <http://arxiv.org/abs/0911.5615>.
- [Su2] Z.W. Sun, *On sums involving products of three binomial coefficients*, preprint, arXiv:1012.3141. <http://arxiv.org/abs/1012.3141>.
- [Su3] Z.W. Sun, *On Apery numbers and generalized central trinomial coefficients*, preprint, arXiv:1006.2776. <http://arxiv.org/abs/1006.2776>.
- [Su4] Z.W. Sun, *On sums related to central binomial and trinomial coefficients*, preprint, arXiv:1101.0600. <http://arxiv.org/abs/1101.0600>.
- [Su5] Z.W. Sun, *Conjectures and results on $x^2 \pmod{p^2}$ with $p = x^2 + dy^2$* , preprint, arXiv:1103.4325. <http://arxiv.org/abs/1103.4325>.